MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 4

Next we will define the Mitchell forcing, $\mathbb{M}$. Recall that the Cohen forcing to add κ many reals, $Add(\omega, \kappa)$, consists of partial finite functions from $\kappa \times \omega \to \{0,1\}$. For $\alpha < \kappa$ and $p \in Add(\omega, \kappa)$, $p \restriction \alpha$ will denote $p \restriction \alpha \times \omega$; this is a condition in $Add(\omega, \alpha)$. Also, $Add(\omega_1, 1)$ is the Cohen poset to add one subset of ω_1, i.e. conditions are countable partial functions from $\omega_1 \to \{0,1\}$. We use the following notation:

- $\mathbb{P} := Add(\omega, \kappa)$,
- for $\alpha < \kappa$, $\mathbb{P}_\alpha := Add(\omega, \alpha) := \mathbb{P} \restriction \alpha$.

Conditions in the Mitchell forcing $\mathbb{M}$ are pairs of the form (p, q), such that

1. $p \in \mathbb{P}$,
2. $\mathrm{dom}(q)$ is a countable subset of κ.
3. for all $\alpha \in \mathrm{dom}(q)$, $1_{\mathbb{P}_\alpha} \Vdash q(\alpha) \in \dot{Add}(\omega_1, 1)$.

Lemma 1. *$\mathbb{M}$ projects to $\mathbb{P}$.*

Proof. Let $\pi : \mathbb{M} \to \mathbb{P}$ be $\pi(p, q) = p$. This is a projection. $\square$

Lemma 2. *$\mathbb{M}$ has the κ-c.c., and so preserves cardinals greater than or equal to κ.*

Proof. Suppose that $\{(p_\eta, q_\eta) \mid \eta < \kappa\}$ are conditions in $\mathbb{M}$. By the Δ-system lemma, applied to the domains of each p_η and q_η, there is an unbounded $I \subset \kappa$, a finite $d \subset \kappa \times \omega$, and countable $b \subset \kappa$, such that for every $\eta, \delta \in I$, $\mathrm{dom}(p_\eta) \cap \mathrm{dom}(p_\delta) = d$ and $\mathrm{dom}(q_\eta) \cap \mathrm{dom}(q_\delta) = b$.

There are only finitely many possibilities for $p_\eta \restriction d$, so by taking another unbounded $J \subset I$, we may assume that for all $\eta, \delta \in J$, $p_\eta \cup p_\delta$ is a well defined function. Also, for every $\eta \in J$ and $\gamma \in b$, $q_\eta(\gamma)$ is a $\mathbb{P}_\gamma$ name for an element in $Add(\omega_1, 1)$. Since $|P_\gamma| < \kappa$, there are less than κ many possibilities for the value of $q_\eta(\gamma)$. So, there are $\eta < \delta$, both in J, such that (p_η, q_η) and (p_δ, q_δ) are compatible.

$\square$

Lemma 3. *$\mathbb{M}$ projects to $Add(\omega, \alpha) * \dot{Add}(\omega_1, 1)$*

Proof. Let $\pi : \mathbb{M} \to Add(\omega, \alpha) * \dot{Add}(\omega_1, 1)$ be $\pi(p, q) = (p \restriction \alpha, q(\alpha))$. We claim that this is a projection. If $(p', q') \leq_{\mathbb{M}} (p, q)$, then $p' \restriction \alpha \leq_{\mathbb{P}_\alpha} p \restriction \alpha$, and if $\alpha \in \mathrm{dom}(q) \subset \mathrm{dom}(q')$, then $p' \restriction \alpha \Vdash q'(\alpha) \leq q(\alpha)$. (If $\alpha \notin \mathrm{dom}(q)$, then $q'(\alpha) = q(\alpha) = \emptyset = 1_{\dot{Add}(\omega_1, 1)}$). So, π is order preserving.

For the second requirement of being a projection, suppose that $(r, s) \leq \pi(p, q) = (p \restriction \alpha, q(\alpha))$. Let $p' = p \cup r$. Since $r \leq p \restriction \alpha$, p' is a well defined function, and so $p' \in \mathbb{P}$. Let q' be such that $\mathrm{dom}(q) = \mathrm{dom}(q')$;

$q'(\alpha) = s$ and for $\beta \in \mathrm{dom}(q) \setminus \{\alpha\}$, $q'(\beta) = q(\beta)$. Then $(p', q') \leq_{\mathbb{M}} (p, q)$ and $\pi(p', q') = (r, s)$. $\square$

Lemma 4. *$Add(\omega, \alpha) * \dot{Add}(\omega_1, 1)$ collapses α to ω_1.*

Proof. Let G be $Add(\omega, \alpha)$-generic. In $V[G]$, let $\langle r_i \mid i < \alpha \rangle$ be distinct elements of 2^ω. Let H be $Add(\omega_1, 1)$-generic over $V[G]$. In $V[G][H]$ define $h : \alpha \to \omega_1$, by setting $h(i)$ to be the least limit $\beta < \omega_1$ such that there is $p \in H$ with:

- $\beta + n \in \mathrm{dom}(p)$ for all n, and
- for every $n < \omega$, $p(\beta + n) = r_i(n)$.

Then h is a one-to-one function from α to ω_1.

$\square$

Corollary 5. *$\mathbb{M}$ collapses every uncountable $\alpha < \kappa$ to ω_1.*

Definition 6. $\mathbb{Q} := \{q \mid \mathrm{dom}(q) \subset \kappa, |\mathrm{dom}(q)| < \omega_1, (\forall \alpha \in \mathrm{dom}(q)) 1_{\mathbb{P}_\alpha} \Vdash q(\alpha) \in \dot{Add}(\omega_1, 1)\}$ *(i.e. the second coordinates of conditions in $\mathbb{M}$). For $q_1, q_2 \in \mathbb{Q}$, we set $q_2 \leq_{\mathbb{Q}} q_1$ iff:*

- $\mathrm{dom}(q_2) \supset \mathrm{dom}(q_1)$,
- *for all $\alpha \in \mathrm{dom}(q_1)$, $1_{\mathbb{P}_\alpha} \Vdash q_2(\alpha) \leq q_1(\alpha)$.*

Next we consider the product $\mathbb{P} \times \mathbb{Q}$. This has the same underlying set as $\mathbb{M}$, but the ordering is different. For example note that $(p', q') \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$ implies that $(p', q') \leq_{\mathbb{M}} (p, q)$, but the converse fails. Actually, $(p', q') \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$ iff $p' \leq_{\mathbb{P}} p$ and $(1, q') \leq_{\mathbb{M}} (1, q)$

Lemma 7. *$\mathbb{P} \times \mathbb{Q}$ projects to $\mathbb{M}$.*

Proof. Let $\pi : \mathbb{P} \times \mathbb{Q} \to \mathbb{M}$ be the identity, i.e. $\pi(p, q) = (p, q)$. We will show that this is a projection. Suppose that $(p', q') \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$. That means that $p' \leq_{\mathbb{P}} p$ and $q' \leq_{\mathbb{Q}} q$. The latter implies that $\mathrm{dom}(q') \supset \mathrm{dom}(q)$ and for every $\alpha \in \mathrm{dom}(q)$, $1_{\mathbb{P}_\alpha} \Vdash q'(\alpha) \leq_{\dot{Add}(\omega_1,1)} q(\alpha)$. Therefore, $p' \restriction \alpha \Vdash q'(\alpha) \leq_{\dot{Add}(\omega_1,1)} q(\alpha)$, and so $(p', q') \leq_{\mathbb{M}} (p, q)$. So π is order preserving.

To show the second requirement of being a projection, suppose that $(p', q') \leq_{\mathbb{M}} (p, q) = \pi(p, q)$. That means that $p' \leq_{\mathbb{P}} p$; $\mathrm{dom}(q) \subset \mathrm{dom}(q')$ and for every $\alpha \in \mathrm{dom}(q)$, $p' \restriction \alpha \Vdash q'(\alpha) \leq q(\alpha)$. We have to define a condition $(r, s) \in \mathbb{P} \times \mathbb{Q}$, such that:

- $(r, s) \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$, and
- $\pi(r, s) = (r, s) \leq_{\mathbb{M}} (p', q')$.

Set $r = p'$. Let $s \in \mathbb{Q}$ be such that $\mathrm{dom}(s) = \mathrm{dom}(q')$. For every $\alpha \in \mathrm{dom}(q)$, we define $s(\alpha)$ to be a $\mathbb{P}_\alpha$-name for $Add(\omega_1, 1)$, such that $p' \restriction \alpha \Vdash s(\alpha) = q'(\alpha)$ and if $t \in \mathbb{P}_\alpha$ is incompatible with $p' \restriction \alpha$, then $t \Vdash_{\mathbb{P}_\alpha} s(\alpha) = q(\alpha)$. We can always cook up such a name [1]. Then we have:

(1) $1_{\mathbb{P}_\alpha} \Vdash s(\alpha) \leq q(\alpha)$
(2) $p' \restriction \alpha \Vdash s(\alpha) \leq q'(\alpha)$.

[1] Formally, $s(\alpha) = \{\langle \sigma, t \rangle \mid (t \leq p' \restriction \alpha, t \Vdash \sigma \in q'(\alpha))$ or $t \perp p' \restriction \alpha, t \Vdash \sigma \in q(\alpha)\}$.

The first item guarantees that $(p', s) \leq_{\mathbb{P}\times\mathbb{Q}} (p, q)$, and the second item gives that $(p', s) \leq_{\mathbb{M}} (p', q')$. That concludes the proof that π is a projection.

$\square$

Lemma 8. $\mathbb{Q}$ *is* ω_1*-closed.*

Proof. Suppose that $\langle q_n \mid n < \omega\rangle$ is a decreasing sequence of conditions in $\mathbb{Q}$. Define a lower bound q as follows. Set $\mathrm{dom}(q) = \cup_n \mathrm{dom}(q_n)$. This is still countable. For every $\alpha \in \mathrm{dom}(q)$, let n_α be such that $\alpha \in \mathrm{dom}(q_{n_\alpha})$. Since the sequence is decreasing, $\alpha \in \mathrm{dom}(q_n)$ for all $n \geq n_\alpha$ and:

- $1_{\mathbb{P}_\alpha} \Vdash$ "$\langle q_n(\alpha)_| n_\alpha \leq n < \omega\rangle$ is a decreasing sequence in $\dot{Add}(\omega_1, 1)$";
- $1_{\mathbb{P}_\alpha} \Vdash$ "$\dot{Add}(\omega_1, 1)$ is ω_1-closed.

It follows that $1_{\mathbb{P}_\alpha} \Vdash$ "$(\exists x \in \dot{Add}(\omega_1, 1))(\forall n \geq n_\alpha)(x \leq q_n(\alpha))$. Then there is a $\mathbb{P}_\alpha$-name, s, such that $1_{\mathbb{P}_\alpha} \Vdash$ "$s \leq_{\dot{Add}(\omega_1,1)} q_n(\alpha)$" for all $n \geq {n_\alpha}$[2]. Set $q(\alpha) = s$.

$\square$

Easton lemma: Suppose that $\mathbb{P}$ is τ-c.c. and $\mathbb{Q}$ is τ-closed. Then $1_{\mathbb{P}} \Vdash \mathbb{Q}$ does not add any new sequences of length less than τ.

Lemma 9. $\mathbb{M}$ *preserves* ω_1

Proof. Let $G_{\mathbb{M}}$ is $\mathbb{M}$-generic and $G_{\mathbb{P}} \times G_{\mathbb{Q}}$ be $\mathbb{P} \times \mathbb{Q}$-generic, such that $V[G_{\mathbb{M}}] \subset V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$. Since $\mathbb{P}$ has the countable chain condition, ω_1 is still a cardinal in $V[G_{\mathbb{P}}]$. By Easton's lemma, every countable sequence from $V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$ is actually in $V[G_{\mathbb{P}}]$. So ω_1 remains a cardinal in $V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$. (Otherwise there would have been a new countable sequence $\omega \to \omega_1$).

Since $V[G_{\mathbb{M}}] \subset V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$, ω_1 is a cardinal in $V[G_{\mathbb{M}}]$. $\square$

Theorem 10. *If* $G_{\mathbb{M}}$ *is* $\mathbb{M}$*-generic, then* $V[G_{\mathbb{M}}] \models \kappa = 2^\omega = \omega_2$.

Proof. We already showed that ω_1 and κ remain cardinals, while everything in between is collapsed. It follows that $V[G_{\mathbb{M}}] \models \kappa = \omega_2$. Also since the forcing projects to $Add(\omega, \kappa)$, $V[G_{\mathbb{M}}] \models \kappa = 2^\omega$. $\square$

[2]This is due to the fact that if $p \Vdash (\exists x)\phi(x)$, then there is a name a, such that $p \Vdash \phi(a)$. The proof is a good exercise on forcing.